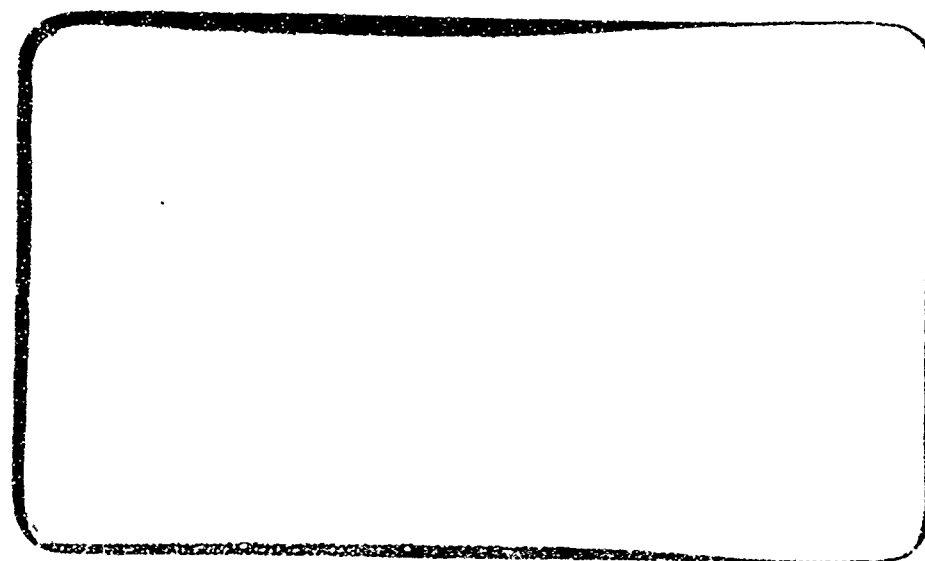


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THEORY AND APPLICATIONS OF THE  
NOTION OF COMPLEX SIGNAL

J. Ville

Translated from the French by I. Selin

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THEORY AND APPLICATIONS OF THE NOTION OF COMPLEX SIGNAL

by J. Ville

ABSTRACT

The present article is a contribution to the problem of the composite representation of a signal by a two-dimensional distribution of energy in a domain defined by two axes, the time axis and the frequency axis. The author proposes such a distribution, using operators analogous to those used in quantum mechanics. He thus obtains a definition of the instantaneous spectrum of a signal, and of the distribution of the energy corresponding to one frequency. By integration (with respect to time) of the instantaneous spectrum (which varies with time) the spectrum, in the usual sense of the word, is recovered. The author defines the instantaneous frequency of a signal in the same way, using the notion of a complex signal (obtained by the complex extension of the real signal when time is considered as a complex variable). These notions of instantaneous frequency and of the instantaneous spectrum are introduced to furnish a firm theoretical basis for studies of frequency modulation, of continuous harmonic analysis, of frequency compression, and, in a general way, of all the problems for which classical harmonic analysis furnishes a description which departs too far from physical reality.

Tr. Note:

The term "Analytic Signal" is often used where "Complex Signal" here appears.

PLAN

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#### THE PRINCIPAL SYMBOLS USED

$t$	= time
$s(t)$	= real signal
$\sigma(t)$	= quadrature signal
$\psi(t)$	= $s(t) + j\sigma(t)$ : complex signal
$f$	= frequency (in cycles)
$\omega$	= frequency (in radians)
$s(\omega)$	= spectrum of a signal (frequency expressed in radians)
$\phi(f)$	= spectrum of a signal (frequency expressed in cycles)
$\bar{t}$	= mean value of time
$\bar{f}$	= mean frequency
$t_f$	= mean delay associated with frequency $f$
$f_t$	= instantaneous frequency at the time $t$
$j$	= imaginary unit
$z^*$	= $z$ - conjugate
$\bar{u}$	= mean value of $u$
$ u $	= modulus of $u$

# I. BIBLIOGRAPHICAL NOTE

The fundamental articles which deal with the general theory of variable frequency circuits are:

J. R. Carson and T. C. Fry, Variable Frequency Electric Circuit Theory, Bell System Technical Journal, 1937, t. XVI, p.513

D. Gabor, Theory of Communication, Journal of the Institution of Electrical Engineers, vol. 93, part III, no. 26, Nov. 1946, p. 429

In the first of these articles is studied the behavior of a signal of the form  $\text{Cos} \left[ \int \omega(t) dt + \varphi \right]$ , a function of the variable frequency  $\omega(t)$ , in passing through a network. In the second, the author decomposes a signal into a double series of signals, where each of the elementary signals occupies a certain domain of the two-dimensional time-frequency diagram. Carson, Fry, and Gabor have also considered composite representations, expanding the signal in time and on the frequency scale at the same time, but without giving an exact definition of the distribution of energy in a similar diagram, which leaves a certain arbitrariness in the methods of representation that they choose; on the contrary, in the present work we study a two-dimensional distribution of the energy, which assumes nothing about the form of the elementary signals used to analyse the given signal.

## II. INTRODUCTION

### QUANTITY OF INFORMATION TRANSMITTED AND THE COMPLEXITY OF A SIGNAL

The transmission of communication signals is accomplished by means of a transmission of energy, generally of electromagnetic or of acoustic energy. In contrast to the case of power transmission, it is not energy itself which is of interest, but rather the changes in this energy in the course of time. The more complicated the function which represents, as a function of time, the change in voltage, current, pressure, or any other carrier, the greater is the amount of information carried by the transmitted energy.

### EVALUATION OF THE COMPLEXITY OF A SIGNAL BY THE NUMBER OF APPRECIABLE HARMONICS WHICH ARE CONTAINED IN ITS FOURIER EXPANSION

In practice, in order to evaluate the degree of complexity of a function in a transmission system, one proceeds to harmonic analysis; that is, one expands the function into sinusoidal components in the form of a Fourier series or integral. A function  $s(t)$  may therefore be considered from two different points of view:

1. From the first point of view, the function is considered directly; to each value of  $t$  there is associated a value of  $s$ , and complication of the function increases with the number of variations shown by the curve which represents  $s$  as a function of  $t$ . In the case of acoustic signals, this first point of view is difficult to specify; on the other hand, in the case of telegraph signals, which are composed of a series of pulses of the same length and height, it is easily seen that the function becomes more complicated directly with the increase of pulses which occurs in a given interval of time.

2. From the second point of view, the function is considered to be composed of the superposition of sinusoidal functions which differ in amplitude and in phase. The function becomes more complicated directly as the number of sinusoidal components of appreciable amplitude increases. The second point of view is of interest in the study of distortions; in fact, if the signal is transmitted across a distorting system which suppresses certain sinusoidal components, there is a certain loss of information in this deformation which may be evaluated. From this it is seen that the capacity of a communication channel may be evaluated; this quantity being proportional to the number of independent frequencies that the channel can carry. The consideration of the second point of view is essential in communication theory.

#### THE NECESSITY OF COMPOSITE EXPANSIONS IN TIME AND FREQUENCY FOR THE STUDY OF CERTAIN QUESTIONS

There exist some questions where the preceding points of view are insufficient. Since it is desirable to use a communication channel to the maximum, one is led to renounce the integral transmission of a signal; anyway, distortions are inevitable. In particular, it is known that if the physiological impression produced by the superposition of harmonics were not independent of the relative phase of these harmonics to a certain degree, no long-distance telephone transmission would be possible. In the same way, frequency compression would be physically unrealizable if the relative phase shifts of the different components had to be maintained in the compression of the frequencies. But this tolerance in transmission is not admissible in certain cases. If we consider a continuous note, emitted by an organ, for instance, we can



shift the phase of the harmonics without changing the sound. Consider a symphony; if we expanded the corresponding sound in a Fourier series, we would obtain a series of harmonics, heard for the duration of the symphony, whose phases we obviously may not arbitrarily change.

So we see the necessity of going past the harmonic analysis of a function. We are going to show what general course we should follow in the particular case of an acoustic signal.

#### PRELIMINARY REMARKS ON THE CONCEPT OF INSTANTANEOUS SPECTRUM

On hearing a fragment of music, there is no connection between the physiological impression received at an instant, and the amplitude of the acoustic signal  $s(t)$  considered: the ear reacts only to a succession of values of  $s(t)$ . But this does not at all imply that the definition of  $s(t)$  in terms of its sinusoidal components is perfect. If we consider a fragment containing many measures (which is the least that one should ask) and one note, la for example, appears once in the fragment, the harmonic analysis will present us with the amplitude and the phase of the corresponding frequency, without locating the la in time. Now then, it is obvious that in the course of the fragment there will be instants when the la will not be heard. Nevertheless, the representation is mathematically correct, because the phase of the notes near la acts to destroy this note by interference when la is not heard, and to reinforce it, also by interference, when it is heard; but if there exists in this concept a cleverness which does honor to mathematical analysis, there is also a distortion of reality; in fact, when la is not heard, the true reason is that la is not emitted.

It is therefore desirable to search for a composite definition of the signal, of the kind anticipated by Gabor: at each instant a number of frequencies is presented, giving the strength and the pitch of the sound which

is heard; with each frequency there is associated a certain distribution in time which defines the intervals during which the corresponding note is emitted. This leads to the definition of the instantaneous spectrum as a function of time, which gives the structure of a signal at a given instant; the spectrum of the signal, in the usual sense of the term, which gives the frequency composition for the whole time interval of emission, is obtained by summing all the instantaneous spectra (in a rigorous way, by integrating with respect to time). In correlating, one is led to the distribution of frequencies in time; the signal is recovered by integrating these distributions.

Unfortunately, things are not as simple as they seem at first glance. We actually see that we have to envision the continuous harmonic analysis of a signal; for such an analysis we can:

1. First cut the signal into pieces (in time) by a commutator:  
then present these different slices to a system of filters to analyse them, or
2. First filter the different frequency bands: then cut the bands into pieces (in time) to study the energy variations.

Continuous harmonic analysis consists then of the application of two operators (filtering and commutation).

But these two operators are such that either one of them, if it is very exact (very short pieces or very narrow bands) renders the other inoperative, because it deforms the signal considerably (by the introduction of transient currents for commutation, of a long duration for narrow band filters). The instantaneous spectrum may only be determined, physically, to some approximation. But another question is presented us: approximation to what? That is why we have tried to obtain a precise definition of instantaneous spectrum,

in order to have a theoretical basis, non-existent until the present, to guide research into the specification of apparatus for continuous harmonic analysis, frequency compression, or any other techniques for which the classical concepts of frequency and spectrum are insufficient.

After these general considerations, we shall pose the question in more precise terms.

# GENERAL STATEMENT OF THE PROBLEM OF INSTANTANEOUS SPECTRUM AND INSTANTANEOUS FREQUENCY

The difficulties presented by the definition of instantaneous frequency of a signal are well known. These difficulties stem from the fact that the frequency of a sinusoid is defined rigorously only when it exists for an infinite duration; the spectrum of a signal  $s(t)$ , which may be expressed as

$$(1) \quad S(\omega) = \frac{1}{2\pi} \int s(t) e^{-j\omega t} dt$$

is defined for the signal ensemble,  $s$  does not contain time explicitly. Therefore, in the classical theory, neither the instantaneous spectrum at the instant  $t$  nor the instantaneous frequency are susceptible to definition. This still concerns primitive concepts. If, for example, we consider a low frequency modulated signal of the form

$$(2) \quad s(t) = \cos(\omega_0 t + \frac{\Delta\omega}{\Omega} \sin \Omega t)$$

it is evident that the "instantaneous frequency" (which is conventionally defined in the case of frequency modulation)

$$(3) \quad \omega = \frac{d}{dt} (\omega_0 t + \frac{\Delta\omega}{\Omega} \sin \Omega t) = \omega_0 + \Delta\omega \cos \Omega t$$

has a physical significance, which grows precise as  $\Omega$  decreases with respect to  $\omega_0$ .

We here propose to give a definition of instantaneous spectrum, valid for a fairly extensive class of signals, to develop certain applications of the considerations to which this problem will lead us. The definition which we propose to make introduces conventions which may appear arbitrary, but which are justified by the coherence of the results and by the parallelism with the analogous conventions which have proved so fertile in quantum mechanics.

Our point of departure is the following: it has been easy to associate an instantaneous frequency with the signal (2) because this signal "may be considered" as the real part of the signal

$$(4) \quad \psi(t) = e^{j(\omega_0 t + \frac{\Delta\omega}{\Omega} \sin \omega t)}$$

which has a constant modulus. The instantaneous frequency, in radians per second, is merely

$$(5) \quad \omega = \frac{d}{dt} \arg \psi$$

We need only, in the case of any signal, place it in the form

$$(6) \quad s(t) = \frac{1}{2} [\psi(t) + \psi^*(t)]$$

in order to extend it into the complex plane, and we shall obtain the instantaneous frequency by expression (5);  $\psi(t)$  will be called the complex signal. The first part of the article will be devoted to the establishment of the corresponding expressions. In order to justify, from another point of view, this definition of instantaneous frequency, we shall attach the concept of group velocity to it, which constitutes a second part. We immediately pass to the concept of instantaneous spectrum, because instantaneous frequency defines a signal only grossly, and it should be considered as

the mean value, at a given instant, of the frequencies of the instantaneous spectrum. We treat this question in Part III, according to the following principles: a signal may be considered as being a certain amount of energy, whose distribution in time (given by the form of the signal) and in frequency (given by the spectrum) is known. If the signal extends through an interval of time  $T$  and an interval of frequencies  $\Omega$ , we have a distribution of energy in a rectangle of area  $T\Omega$ . We know the projections of this distribution upon the sides of the rectangle, but we do not know the distribution in the rectangle itself. If we try to determine the distribution within the rectangle, we run into the following difficulty: If we cut up the signal on the time scale, we display the frequencies; if we cut it up on the frequency scale, we display the times. The distribution cannot be determined by successive measures. A simultaneous determination must be sought, which has only a theoretical significance: therefore we must operate either on the signal or on the spectrum. But for the signal where, for example, time is a variable, frequency is properly speaking an operator (the operator  $(1/2 \pi j)d/dt$ , for frequencies in cps). We have determined the simultaneous distribution of  $t$  and of  $(1/2 \pi j)d/dt$ , by methods of the calculus of probabilities, which easily leads to the instantaneous spectrum (and just as easily to the distribution in time of the energy associated with one frequency). It is seen that the formal character of the method of calculation used is imposed by the nature of the difficulty encountered, which is analogous to that which occurs in quantum mechanics when non-permutable operators must be composed. We shall use many results due to Gabor (Theory of Communication), and the same notation, which allows us to avoid some problems to proceed to the development of new points of view. Finally, Part IV contains some applications of the concept of complex signal which has been imposed upon us, as we have said, in the research into instantaneous frequency.

### III. PART ONE: COMPLEX SIGNAL AND INSTANTANEOUS FREQUENCY

#### EXTENSION OF A REAL SIGNAL IN THE COMPLEX PLANE

Let us consider the signal

$$(1) \quad s(t) = \cos \omega t$$

We may consider it, either as the real part of

$$(2) \quad \psi(t) = e^{j\omega t}$$

or of

$$(3) \quad \psi^*(t) = e^{-j\omega t}$$

Since  $\underline{t}$  takes only real values, there is no reason to choose one of the forms rather than the other. This is not so if  $\underline{t}$  takes the complex values

$$(4) \quad t = \underline{t} + j \theta$$

since now some differences appear. If, in fact,  $\theta$  tends toward  $+\infty$ ,

$\psi$  tends toward zero and  $\psi^*$  toward infinity.

If we decide to retain, insofar as possible, only the functions which are regular in the upper half-planes, we then choose

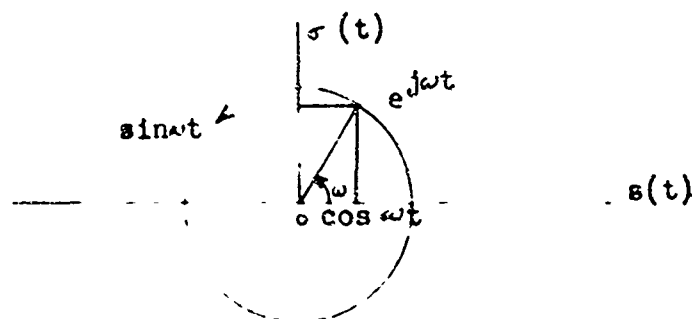


Figure 1. The instantaneous frequency  $\omega$  of a signal  $\cos \omega t$  is the angular velocity of the point of coordinates  $\cos \omega t$ ,  $\sin \omega t$  ( $\sin \omega t$  is the signal in quadrature with  $\cos \omega t$ ).

$$(5) \quad \psi(t) = e^{j\omega t}$$

Let us take another simple example:

$$(6) \quad s(t) = \frac{1}{1+t^2}$$

We may consider it as the real part of either of two functions

$$(6a) \quad \frac{1}{1+jt} \quad \text{or} \quad \frac{1}{1-jt}$$

Only one of these, the second one, is regular in the upper half-plane; therefore we choose

$$(7) \quad \psi(t) = \frac{1}{1-jt}$$

Let us consider a third example:

$$(8) \quad s(t) = \begin{cases} 1 & : -1 < t < +1 \\ 0 & : t < -1 \text{ where } t > 1 \end{cases}$$

We can consider  $s(t)$  as the real part of

$$(9) \quad \psi(t) = \frac{1}{\pi j} \log \frac{t-j}{t+j}$$

with a conveniently chosen determination. Function (9) is not uniform; we make it uniform in the upper half-plane by excluding the singular points  $\pm j$ .

In general, the principles above lead to the association of the quadrature signal

$$(10) \quad \sigma(t) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{s(\tau) d\tau}{\tau - t}$$

and the function

$$(11) \quad w(t) = s(t) + j\sigma(t) = \frac{1}{\pi j} \int_{-\infty}^{+\infty} \frac{s(\tau) d\tau}{t - \tau}$$

with  $s(t)$ .

Equation 11 defines  $w$  as a function of the complex variable  $t$ , holomorphic in the upper half-plane. It is known that, conversely, for  $t$  real:

$$(12) \quad s(t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{s(\tau) d\tau}{\tau - t}$$

COMPLEX SIGNALS. THEY MAY BE CONSIDERED AS THE RESULT OF THE MODULATION OF THEIR ENVELOPE BY A CARRIER WHICH IS ITSELF FREQUENCY MODULATED. INSTANTANEOUS FREQUENCY.

A signal such as  $w(t)$  will be called a complex signal. We can define  $s(t)$  by

$$(13) \quad s(t) = \text{Re} \left[ |w(t)| e^{j \arg w} \right]$$

which defines  $s(t)$  as the result of the modulation of the signal  $e^{j \arg w}$  by the signal  $|w|$ . If we consider  $|w|$  as representing the envelope of  $s(t)$ , and  $e^{j \arg w}$  as a frequency modulated signal, we obtain for the instantaneous frequency (Fig. 2)

$$(14) \quad f_t = \frac{1}{2\pi} \frac{d}{dt} \arg w = \frac{1}{4\pi j} \frac{w^*}{w} \frac{d}{dt} \frac{w}{w^*}$$

For  $s(t) = \cos(\omega t + \phi)$ , we obtain

$$(15) \quad \begin{aligned} w(t) &= e^{j(\omega t + \phi)} \\ f_t &= \frac{\omega}{2\pi} \end{aligned} \quad \omega > 0$$



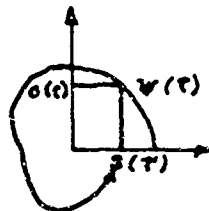


Fig. 2 -- The instantaneous frequency of the signal  $s(t)$  is the angular velocity of the point  $\psi(t) = s(t) + j\sigma(t)$ , where  $\sigma(t)$  is the signal in quadrature with respect to  $s(t)$

The instantaneous frequency is constant for this signal. If we consider the signal:

$$(15a) \quad s(t) = \frac{1}{1 + t^2} = R_e \frac{1}{1 - jt}$$

we obtain:

$$(16) \quad t = \frac{1}{2\pi} \frac{1}{1 + t^2}$$

Let us now consider a modulated sinusoidal signal:

$$(16a) \quad s(t) \cos \omega t \cos \Omega t \quad \Omega > \omega > 0$$

then we have

$$(16b) \quad \psi(t) = \frac{1}{2} \left[ e^{j(\Omega + \omega)t} + e^{j(\Omega - \omega)t} \right]$$

$$|\psi(t)| = \cos \omega t \quad f_t = \frac{\Omega}{2\pi}$$

The instantaneous frequency is the carrier frequency, and  $|\psi|$  is the envelope as it is usually defined. Note the importance of the fact that the coefficients  $t$  of the exponentials are positive. Neglect of this point would lead to the inversion of the roles of  $\omega$  and  $\Omega$ , and would produce absurd results.

ANY SIGNAL MODULATED BY A SUFFICIENTLY HIGH FREQUENCY MAY BE CONSIDERED COMPLEX.

The fact that in the preceding example the carrier frequency turned out to be the instantaneous frequency is not an isolated coincidence, but rather results from the following proposition:

For any signal  $s(t)$ , the function

$$(a) \quad \bar{\psi}(t) = s(t) e^{j\omega_0 t} \quad \omega_0 > 0$$

which in general is not complex, approaches the complex signal  $\psi(t)$  [associated with  $s(t) \cos \omega_0 t$ ] as  $\omega_0$  increases.

Consequently, the instantaneous frequency  $\omega_0/2$  may be attributed to  $s(t) \cos \omega_0 t$ , for very large values of  $\omega_0$ .

The proposition which we shall come to use is itself an immediate consequence of the fact that a complex signal is characterized by the peculiarity of having a spectrum whose amplitude is zero for negative frequencies. Now, modulating  $s(t)$  by  $e^{j\omega_0 t}$  amounts to causing the spectrum of  $s(t)$  to be translated by the amount  $\omega_0$ . For a large enough value of  $\omega_0$ , the spectrum lies entirely in the region of positive frequencies, and  $s(t) e^{j\omega_0 t}$  becomes complex.<sup>1</sup>

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<sup>1</sup>This is the real reason that Gabor uses bell-shaped modulated signals.

#### IV. PART TWO: GROUP VELOCITY AND INSTANTANEOUS FREQUENCY

##### GROUP AND PHASE VELOCITY

It is well known that if a signal has a phase velocity  $V_\phi$ , its group velocity is

$$(1) \quad V_g = \frac{d\omega}{d(\omega/V_\phi)}$$

It is shown in this formula that the simple characteristics are the inverses of the group and phase velocities, i.e., group and phase propagation delays for a given distance.

##### GROUP DELAY OF A SIGNAL CONSIDERED AS THE WEIGHTED MEAN OF THE GROUP DELAY OF DIFFERENT FREQUENCIES

Let us now consider a complex signal  $\psi_t$ , and let  $\phi_t$  be its spectrum;<sup>1</sup> then we have:

$$(2) \quad \psi(t) = \int \phi(f) e^{2\pi jft} df$$

$$(3) \quad \phi(f) = \int \psi(t) e^{-2\pi jft} dt$$

$$(4) \quad (-1)^n \int \psi^* t^n \psi dt = \int \phi^* \left[ \frac{1}{2\pi j} \frac{d}{df} \right]^n \phi df$$

$$(5) \quad \int \phi^* f^n \phi df = \int \psi^* \left[ \frac{1}{2\pi j} \frac{d}{dt} \right]^n \psi dt$$

Let us try to evaluate the delay of the signal. We may consider the energy of the signal as being expressed in the density  $\psi^* \psi$  during the interval  $dt$ . For the delay this gives us the mean value:

---

<sup>1</sup>We shall employ, according to the situation, the notation  $\omega$  or  $f$  for frequency, measured in rad/sec or cps, respectively. The corresponding spectra are  $s(\omega)$  and  $\phi(f)$ .

$$(6) \quad \bar{t} = \frac{\int \psi^* t \psi dt}{\int \psi^* \psi dt}$$

Since this delay was determined by energy considerations, it will be called the group delay. Now using Eq. 4, we obtain:

$$(7) \quad \bar{t} = - \frac{1}{2\pi j} \frac{\int \phi^* \frac{d}{df} \phi df}{\int \phi^* \phi df}$$

Let us examine the modulus and the argument of  $\phi$ :

$$(7a) \quad \frac{1}{\phi} \frac{d}{df} \phi = \frac{1}{|\phi|} \frac{d}{df} \phi + j \frac{d}{df} \arg \phi$$

Since  $\bar{t}$  is real, we obtain:

$$(8) \quad \bar{t} = \frac{1}{2\pi} \frac{\int \phi^* \phi \frac{d}{df} \arg \phi df}{\int \phi^* \phi df}$$

Let us consider now that the energy of the signal is distributed on the frequency scale with a density  $\phi\phi^*$ ; we see that  $\bar{t}$ , the group delay with respect to the signal as a whole, is obtained by taking the weighted mean of the quantities:

$$(9) \quad \bar{t}_f = - \frac{1}{2\pi} \frac{d}{df} \arg \phi$$

which we consider as the group delays of the different frequencies. We see that the power associated with a frequency  $f$  may be considered as being present at the mean instant  $\bar{t}_f$ .

#### MEAN FREQUENCY OF A SIGNAL CONSIDERED AS A WEIGHTED MEAN OF DIFFERENT INSTANTANEOUS FREQUENCIES

Let us now reverse the role of  $f$  and  $t$ . We see that the mean frequency of the signal, defined by

$$(a) \quad \bar{f} = \frac{\int \phi^* f \phi df}{\int \phi^* \phi df}$$

may be defined equally well by:

$$(b) \quad \bar{f} = \frac{1}{2\pi} \frac{\int \psi^* \psi \frac{d}{dt} \arg \phi dt}{\int \psi^* \psi dt}$$

that is, as the weighted mean of an instantaneous frequency:

$$(10) \quad f_t = \frac{1}{2\pi} \frac{d}{dt} \arg \psi \quad (\text{in cycles per/second})$$

We shall come across this expression again.

# V. PART THREE: DISTRIBUTION OF ENERGY IN THE TIME FREQUENCY-DIAGRAM

## CHARACTERISTIC FUNCTION OF THE ENERGY DISTRIBUTION

We shall now define the distribution of the energy in time and in frequency, of the form: (Fig. 3)

$$(1) \quad p(t, f) dt df$$

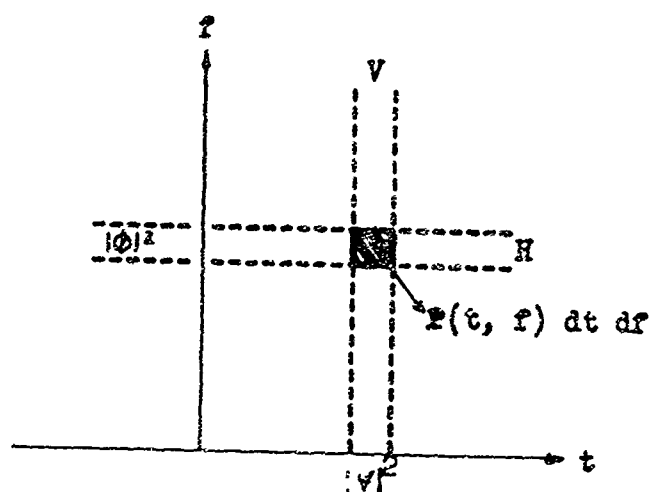


Fig. 3--To the element  $dt df$ , in the time-frequency diagram, corresponds an energy  $p(t, f) dt df$ . The distribution of these energies in a vertical strip gives the instantaneous spectrum at an instant  $t$ ; in a horizontal strip we obtain the energy carried by one frequency as a function of time. By projection on the axes, we get the signal (time axis) and its spectrum (frequency axis).

We shall suppose that, in the following, we are considering a normalized signal, for which:

$$(2) \quad \int \phi^* \phi df = \int \psi^* \psi dt = 1$$

which does not at all restrict the generality.  $p$  possesses the properties of a probability distribution function, and we shall determine its characteristic function:

$$(3) \quad F(u, v) = \iint e^{j(ut + vf)} p(t, f) dt df$$

We have to calculate the mean value of  $e^{j(ut + vf)}$ , but to do this we have only the function  $\psi$ , which does not contain  $f$ , and the function  $\varphi$ , which does not contain  $t$ . We must then consider either  $f$  or  $t$  as an operator. If we work with  $\psi$ , we will arrive at the equation

$$(4) \quad F(u, v) = \int \psi^* e^{j\left(ut + \frac{v}{2\pi j} \frac{d}{dt}\right)} \psi(t) dt$$

Let us see what results from this exponential operator, applied to  $\psi$ . If we split the expression into two parts, we obtain, according to the order in which we consider the factors:

$$(4a) \quad e^{jut} \exp \frac{v}{2\pi} \frac{d}{dt} \psi(t) = e^{jut} \psi\left(t + \frac{v}{2\pi}\right)$$

or:

$$(4b) \quad \exp \frac{v}{2\pi} \frac{d}{dt} e^{jut} \psi(t) = \exp ju\left(t + \frac{v}{2\pi}\right) \psi\left(t + \frac{v}{2\pi}\right)$$

These equations are unacceptable, being incompatible with the relation:

$$(4c) \quad F^*(u, v) = F(-u, -v)$$

which follows from (3) (for real  $u, v$ ).

Let us consider the geometric mean<sup>1</sup> of the two results given by the preceding equations;

---

<sup>1</sup>The form (5) of the operator considered is that which results from the power series expansion of

$$\left(ut + \frac{v}{2\pi j} \frac{d}{dt}\right)^n$$

maintaining the order of the terms.

$$(5) \quad e^{j \left( ut + \frac{v}{2\pi} \frac{d}{dt} \right)} \cdot \psi(t) = e^{ju} \left( t + \frac{v}{4\pi} \right) \cdot \psi \left( t + \frac{v}{2\pi} \right)$$

We obtain (with a slight change in notation):

$$(6) \quad F(u, v) = \int \psi^* \left( t - \frac{v}{4\pi} \right) \cdot \psi \left( t + \frac{v}{4\pi} \right) \cdot e^{jut} dt$$

which is an acceptable form of  $F(u, v)$ .

DISTRIBUTION OF ENERGY. INSTANTANEOUS SPECTRUM, THE SECOND MOMENT OF THE ENERGY ASSOCIATED WITH ONE FREQUENCY AS A FUNCTION OF TIME.

The presence of the factor  $e^{jut}$  indicates that the characteristic function of  $f$  when  $t$  is known is none other than:

$$(7) \quad \psi^* \left( t - \frac{v}{4\pi} \right) \cdot \psi \left( t + \frac{v}{4\pi} \right) / |\psi|^2$$

from which the distribution function of  $f$  when  $t$  is known:

$$(8) \quad p_t(f) = \frac{1}{2\pi|\psi|^2} \int \psi^* \left( t - \frac{v}{4\pi} \right) \psi \left( t + \frac{v}{4\pi} \right) e^{-jvf} dv$$

Since the distribution function of  $t$  is  $|\psi(t)|^2$ , we finally obtain:

$$(9) \quad p(t, f) = \frac{1}{2\pi} \int \psi^* \left( t - \frac{v}{4\pi} \right) \psi \left( t + \frac{v}{4\pi} \right) e^{-jvf} dv$$

The modulus of the instantaneous spectrum<sup>1</sup> is given by Eq. 8; in an analogous manner, the distribution of the energy associated with one frequency would be

$$(10) \quad p_f(t) = \frac{1}{2\pi|\psi|^2} \int \psi^* \left( t + \frac{u}{4\pi} \right) \psi \left( t - \frac{u}{4\pi} \right) e^{-jut} du$$

<sup>1</sup>It does not seem possible to compute the phase of the instantaneous spectrum.



# VI. PART FOUR: APPLICATIONS OF THE CONCEPT OF COMPLEX SIGNAL

The duality which exists between time and frequency permits the clarification of certain phenomena; we shall treat some typical examples.

## DISPLAY OF THE SIGNAL AS A FUNCTION OF TIME. AMPLITUDE AND PHASE DISTORTION

Let  $\psi(t)$  be a complex signal. If we consider the frequency, we see that the signal has an envelope  $|\psi(t)|$  which modulates a signal with a variable instantaneous frequency. Then in the determination of the frequencies, we should expect two terms to appear, one representing variations of instantaneous frequency, the other, variations of the amplitude envelope. Analytically, suppose that we had taken the mean frequency  $\bar{f}$ , the mean of the instantaneous frequencies, as the frequency origin.<sup>1</sup> Then we have:

$$(1) \quad \bar{f} = \int \psi^* \psi f_t dt = 0$$

The second moment of the frequency will be:

$$(2) \quad \bar{f^2} = \int \psi^* f^2 \psi df$$

and the second moment of the instantaneous frequencies:

$$(3) \quad \bar{f_t^2} = \int \psi^* f_t^2 \psi dt$$

Forming the difference and expressing as a function of

$$(4) \quad \bar{f^2} - \bar{f_t^2} = \frac{1}{4\pi^2} \int \left[ \psi^* \frac{d}{dt^2} \psi + \psi^* \left( \frac{d}{dt} \arg \psi \right)^2 \psi \right] dt$$

Where

$$(4a) \quad \psi = |\psi| e^{j \arg \psi}$$

---

<sup>1</sup>We still assume that the signal is normalized.

The differentiation produces:

$$\frac{1}{\psi} \frac{d}{dt} \psi = \frac{1}{|\psi|} \frac{d}{dt} |\psi| + j \frac{d}{dt} \arg \psi$$

$$(4b) \quad \frac{1}{\psi} \frac{d^2}{dt^2} \psi - \frac{1}{\psi^2} \left( \frac{d}{dt} \psi \right)^2 = - \frac{1}{|\psi|^2} \left( \frac{d}{dt} |\psi| \right)^2 + \text{etc...}$$

Substituting the expression of  $\frac{1}{\psi} \frac{d^2}{dt^2} \psi$  in (4), and keeping only the real terms,

$$(4c) \quad \overline{r^2} - \overline{r_t^2} = \frac{1}{4\pi^2} \int \left[ \frac{d}{dt} |\psi| \right]^2 dt$$

from which we obtain the final expression for  $\overline{r^2}$

$$(5) \quad \overline{r^2} = \frac{1}{4\pi^2} \int \psi \psi^* \left[ \left( \frac{d}{dt} \arg \psi \right)^2 + \frac{1}{|\psi|} \frac{d}{dt} |\psi|^2 \right] dt$$

The two previous terms are seen to appear, one depending on the second moment of the instantaneous frequencies, and the other on the amplitude variation of the envelope.

If we consider the second moment in time, we obtain, taking the mean time for the time origin:

$$(6) \quad \overline{t^2} = \frac{1}{4\pi^2} \int \psi \psi^* \left[ \left( \frac{d}{dt} \arg \psi \right)^2 + \left( \frac{1}{|\psi|} \frac{d}{dt} |\psi| \right)^2 \right] dt$$

We re-encounter familiar concepts here: in fact, if we assume that the signal was produced by the passage of a Dirac impulse through a filter with the characteristic  $\phi(f)$ , we will find in  $\overline{t^2}$  the two elements of distortion--the first term results from phase distortion and the second from

amplitude distortion. We note that the phase distortion here introduced is a distortion of the group time of propagation (delay distortion).

Introducing logarithms, we obtain for the standard deviations of time and frequency

$$\begin{aligned} \overline{t^2} &= \frac{1}{4\pi^2} \int |\psi^* \frac{d}{dt} \log \psi|^2 dt = \int \left| \frac{d\psi}{dt} \right|^2 dt \\ \overline{f^2} &= \frac{1}{4\pi^2} \int |\psi^* \frac{d}{df} \log \psi|^2 df = \int \left| \frac{d\psi}{df} \right|^2 df \end{aligned} \quad (7)$$

#### TRANSFER ADMITTANCES

Assume a transfer admittance  $Y(j\omega)$ . If  $\omega$  is complex, the decaying currents of the form  $e^{j\omega t}$  have a positive imaginary part. Then  $Y(j\omega)$  is regular in the lower half-plane. Referring to the properties of a complex signal, we see that it is possible to pass from the form of an admittance to the form of a complex signal by the change of variable

$$(8) \quad \omega = -t$$

If  $\psi(t)$  is complex, from it we can deduce a transfer impedance by the equation

$$(9) \quad Y(j\omega) = \psi(-\omega)$$

The impulse admittance associated with  $y(j\omega)$  is

$$(10) \quad B(t) = \frac{1}{2\pi} \int y(j\omega) e^{j\omega t} d\omega$$

Hence the spectrum of  $\psi(t)$  is

$$(11) \quad S(\omega) = \frac{1}{2\pi} \int e^{-j\omega t} \psi(t) dt = B(\omega)$$

$B(t)$  is zero for negative values of  $t$ , and so  $S(\omega)$  is zero for negative values of  $\omega$ . The only new restriction is that  $B(t)$  be real, which entails that  $\omega(t)$  be a symmetric complex signal, i.e.

$$(12) \quad \psi(-t) = \psi^*(t) \quad (\text{for } t \text{ real})$$

Under condition 12, there is a perfect correspondence between transfer admittances and analytic signals, and between spectra and impulse admittances.

#### BAND-PASS FILTERS; EXAMPLES OF PHYSICALLY REALIZABLE ADMITTANCES

The formation of the transfer admittance of a filter is analogous, as we have seen, to the formation of a complex signal. But to form a complex signal it suffices to extract some  $s(t)$  from a signal and to form its spectrum, from which:

$$(13) \quad s(t) = \int S(\omega) e^{j\omega t} d\omega [S(-\omega) = S^*(\omega)]$$

and consequently, upon doubling and keeping only the terms which correspond to positive frequencies, this leads to

$$(14) \quad \psi(t) = 2 \int_0^{\infty} S(\omega) e^{j\omega t} d\omega$$

In order not to cut into this spectrum (by the neglect of the negative frequency terms) a shift to the right by a suitable modulation may be made.

The corresponding procedure for forming a transfer admittance is the following:

$$(15) \quad B_1(t) = \frac{1}{2\pi} \int Y_1(j\omega) e^{j\omega t} d\omega$$

$B_1(t)$  does not disappear for  $t < 0$  (it is this which shows that  $Y_1$  is not physically realizable).  $B_1(t)$  is shifted to the right and cut off at the left such that  $B_1(t) = 0$  for  $t < 0$ .

$$(16) \quad Y(j\omega) = \int_0^{\infty} B(t) e^{-j\omega t} dt$$

If  $T$  is the amplitude of the translation which  $B_1(t)$  undergoes, we obtain:

$$(16a) \quad Y(j\omega) = \int_0^{\infty} B_1(t - T) e^{-j\omega t} dt$$

If  $B_1(t)$  is symmetric, the tail of the signal may be cut off, which results in:

$$(17) \quad Y(j\omega) = \int_0^{2T} B_1(t - T) e^{-j\omega t} dt$$

This last admittance has the advantage of not causing any phase distortion. But there is an amplitude distortion. For instance, let:

$$Y_1 = \begin{cases} 1 & \text{for } \omega_1 < \omega < \omega_2 \text{ and } -\omega_2 < \omega < -\omega_1 \\ 0 & \text{outside the above intervals} \end{cases}$$

$$B_1(t) = \frac{\sin \omega_2 t - \sin \omega_1 t}{\pi(\omega_2 - \omega_1)t}$$

$$(18) \quad Y(j\omega) = \int_0^{2T} \frac{\sin \omega_2(t - T) - \sin \omega_1(t - T)}{\pi(\omega_2 - \omega_1)(t - T)} dt$$

$Y(j\omega)$  has  $j\omega T$  for its argument; there is no phase distortion. In fact, as  $T$  tends toward infinity,  $|Y|$  tends toward the characteristic of an ideal band-pass; but in the neighborhood of  $\omega_1$  and  $\omega_2$  oscillations due to the Gibbs phenomenon are present.

### GAUSSIAN ADMITTANCE CURVES

Gaussian admittance curves avoid this difficulty. It is well known that the signal:

$$(19) \quad s(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$$

has the spectrum

$$(20) \quad S(\omega) = e^{-\frac{\omega^2}{2}}$$

hence, for the signal:

$$(21) \quad s(t) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(t - t_0)^2}{2\sigma^2}}$$

we obtain as the spectrum:

$$(22) \quad S(\omega) = e^{-\frac{\sigma^2 \omega^2}{2}} e^{-j\omega t_0}$$

If we modulate by  $e^{j\omega_0(t - t_0)}$  ( $\omega_0 > 0$ ), we displace the spectrum of  $\omega_0$ , and therefore:

$$(23) \quad \begin{aligned} \psi(t) &= \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(t - t_0)^2}{2\sigma^2}} e^{j\omega_0(t - t_0)} \\ S(\omega) &= e^{-\frac{\sigma^2(\omega - \omega_0)^2}{2}} e^{-j\omega t_0} \end{aligned}$$

For  $\omega_0 > \frac{3}{\sigma}$ ,  $S(\omega)$  may be considered to be situated in the region  $\omega > 0$ .

To the signal:

$$(24) \quad s(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(t-t_0)^2}{2\sigma^2}} \cos \omega_0(t-t_0)$$

there corresponds the spectrum:

$$(25) \quad S(\omega) = e^{-j\omega t_0} \frac{1}{2} \left[ e^{-\frac{\sigma^2(\omega-\omega_0)^2}{2}} + e^{-\frac{\sigma^2(\omega+\omega_0)^2}{2}} \right]$$

For  $t_0 > 3\sigma$ , we can consider  $S(\omega)$ , given by (25), as a transfer admittance. For  $\omega_0 > 3/\sigma$ , we need conserve only the first term in the parentheses in (25). It is easily shown that if the cosine of (24) is replaced by a sine term, (25) is replaced by:

$$(26) \quad S(\omega) = j e^{-j\omega t_0} \frac{1}{2} \left[ e^{-\frac{\sigma^2(\omega-\omega_0)^2}{2}} - e^{-\frac{\sigma^2(\omega+\omega_0)^2}{2}} \right]$$

Suppose we wish to specify the pass-band  $(\omega_1, \omega_2)$ . We will choose  $\sigma > 3$ ,  $t_0 > 3\sigma > 9/\omega_1$ , and we will integrate with respect to  $\omega_0$  from  $\omega_1$  to  $\omega_2$ , which results in (adding the appropriate coefficients):

$$(27) \quad \begin{aligned} Y(j\omega) = S(\omega) &= e^{-j\omega t_0} \frac{\sigma}{\sqrt{2\pi}} \int_{\omega_1}^{\omega_2} e^{-\frac{\sigma^2(\omega-\omega_0)^2}{2}} d\omega_0 \\ s(t) = B(t) &= \frac{1}{\pi} e^{-\frac{(t-t_0)^2}{2\sigma^2}} \frac{\sin \omega_2(t-t_0) - \sin \omega_1(t-t_0)}{t-t_0} \end{aligned}$$

A low-pass filter would be characterized by:

$$Y(j\omega) = e^{-j\omega t_0} \frac{\sigma}{\sqrt{2\pi}} \int_{-\omega_1}^{\omega_1} e^{-\frac{\sigma^2(\omega - \omega_0)^2}{2}} d\omega_0$$

$$(28) \quad B(t) = \frac{1}{\pi} e^{-\frac{(t - t_0)^2}{2\sigma^2}} \frac{\sin \omega_1(t - t_0)}{t - t_0}$$

#### PHASE-SHIFTING FILTER, SHIFTING THE SIGNAL INTO QUADRATURE

If, instead of (25), we start with the spectrum (26), and we integrate between zero and  $\omega_1$ , we obtain a filter of a fairly special kind, characterized by:

$$Y(j\omega) = -j e^{j\omega t_0} \frac{\sigma}{\sqrt{2\pi}} \int_0^{\omega_1} \left[ e^{-\frac{\sigma^2(\omega - \omega_0)^2}{2}} - e^{-\frac{\sigma^2(\omega + \omega_0)^2}{2}} \right] d\omega_0$$

$$(29) \quad s(t) = B(t) = \frac{1}{\pi} e^{-\frac{(t - t_0)^2}{2\sigma^2}} \frac{1 - \cos \omega_1(t - t_0)}{t - t_0}$$

If  $\omega_1$  and  $\sigma$  tend toward infinity, we see that by suppressing the factor  $e^{-j\omega t_0}$ , we obtain a transfer impedance of the form (for real  $\omega$ ):

$$(30) \quad Y(j\omega) = j \text{ Signal } \omega$$



The complex signal:

$$(30a) \quad \psi(t) = s(t) + js(t)$$

is then transformed into:

$$(30b) \quad \sigma(t - t_0) - j s(t - t_0)$$

Therefore,  $s(t)$  is transformed into  $\sigma(t)$ : the filter defined by Eq. 29 has the property of transforming a real signal into the quadrature signal  $\sigma(t)$ , to a certain approximation which improves as the time of propagation through the filter increases.

The filter (29), for large  $\omega_0$  and  $t_0$  causes a distortion not in amplitude but in phase. Its interest lies in the fact that it provides, it seems to us, the simplest theoretical means of causing the signal to appear in quadrature; this signal never occurred previously except as a computational convenience.

#### QUADRATURE SIGNAL AND SINGLE SIDE-BAND TRANSMISSION

We shall recover the quadrature signal by another procedure, based on experience. It is known that the quadrature signal appears as a parasitic signal in single side-band transmission: the considerations developed here permit a clear explanation of the phenomenon.

Consider a real signal  $s(t)$ , which occupies the frequency band from  $-\omega_1$  to  $\omega_1$ . With this signal, let us modulate a carrier  $\cos \Omega t$ , where  $\Omega > \omega_1$ . With this new signal, let us modulate another carrier,  $\cos \Omega t$ , and we shall filter using  $(-\Omega, \Omega)$  as the pass-band. Then we obtain the signal:

$$(31) \quad s(t) \cdot \cos^2 \Omega t = \frac{1}{2} s(t) [1 + \cos 2\Omega t]$$

which, after filtering, results in  $\frac{1}{2} s(t)$ . This is the ideal case of double-side band transmission (suppressed carrier).

Now let us assume that the second carrier is shifted with respect to the first; let  $\cos(\Omega t + \phi)$  be this second carrier. We would obtain the second signal:

$$(32) \quad s(t) \cos \Omega t \cos (\Omega t + \phi) = \frac{1}{2} s(t) [\cos \phi + \cos (2\Omega t + \phi)]$$

which, after filtering, results in  $\frac{1}{2} s(t) \cos \phi$ . The phase shift is transformed into a signal attenuation.

If we operate with a single side band, we must insert a filter between the two modulations, which obliges us to insert  $\psi(t)$ . Let  $S(\omega)$  be the spectrum of  $\psi(t)$ . After the first modulation, the spectrum becomes:

$$(33) \quad \frac{1}{2} [S(\omega - \Omega) + S(\omega + \Omega)]$$

We keep only the lower band, and continue to the second modulation (Fig. 4).

If the second carrier is in phase with the first one, we recover the original spectrum; but if it is not in phase, i.e., if we multiply by

$$(34) \quad \frac{1}{2} [e^{j(\Omega t + \phi)} + e^{-j(\Omega t + \phi)}]$$

we see that the right half of the spectrum is multiplied by  $e^{j\phi}$ , and the left half by  $e^{-j\phi}$ . The final signal has a spectrum

$$(35) \quad \begin{array}{ll} \text{for } \omega > 0 & S_1(\omega) = \frac{1}{4} e^{j\phi} S(\omega) \\ \text{for } \omega < 0 & S_1(\omega) = \frac{1}{4} e^{-j\phi} S(\omega) \end{array}$$

The corresponding complex signal is then:

$$(36) \quad \psi_1(t) = \frac{1}{4} e^{j\phi} \psi(t)$$

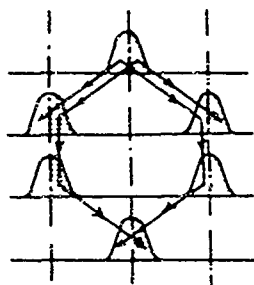


Fig. 4--The spectrum of the signal is shifted to the left and to the right by the first modulation. The filtering suppresses the high frequencies (corresponding to the originally positive frequencies of the right spectrum and the originally negative frequencies of the left spectrum). The second modulation, if it is in phase with the first, restores the original spectrum, putting the positive and negative frequencies back into place. If it is shifted in phase with respect to the first modulation, the positive and negative frequencies are out of phase themselves, whence comes the appearance of a signal in quadrature.

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There is a phase shift of  $\psi(t)$ , and as a consequence a quadrature signal is introduced. We have precisely:

$$(37) \quad s_1(t) = \frac{1}{4} [s(t) \cos \phi - \sigma(t) \sin \phi]$$

We now see the difference between the double side band and single side band modes of transmission.